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Intersection graphs of maximal hypercubes

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Abstract

In this paper we consider cube graphs, that is intersection graphs of maximal hypercubes of graphs. In contrast to the related concepts of line graphs and clique graphs, we show that any graph is a cube graph of a (bipartite) graph. We answer a question of Bandelt and Chepoi (European J. Combin. 17 (1996) 113) by showing that dually chordal graphs are precisely cube graphs of graphs of acyclic cubical complexes. Similarly, we characterize classes of chordal graphs, Helly chordal graphs and doubly chordal graphs as cube graphs of certain subclasses of isometric subgraphs of hypercubes. © 2003 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Characterizations of (natural) classes of graphs can often be obtained via intersections of some family of sets, where vertices of a graph represent sets of a family $\mathcal{F}$ and two vertices are adjacent when corresponding sets are nondisjoint. Well-known classes introduced in this way are interval graphs, circular-arc graphs, chordal graphs, etc. [10, 14]. When $\mathcal{F}$ is a family of subgraphs of a certain type, then we may consider this representation as a transformation of a graph. For example, when $\mathcal{F}$ consists of edges of a graph, we get a *line graph*, and when $\mathcal{F}$ consists of its cliques (maximal complete subgraphs), this is called a *clique graph* of a graph. In this paper we consider the case when $\mathcal{F}$ consists of maximal hypercubes of a graph, and we call it a *cube graph*.

Graphs that are line graphs of graphs have been characterized by forbidden subgraphs and certain partition conditions, cf. [4, 6]. Roberts and Spencer [19] proved a characterization of graphs that are clique graphs of graphs. In particular, this property is shared by *clique-Helly graphs*, the graphs in which cliques enjoy the Helly property (that is, every pairwise nondisjoint family of cliques has a nonempty intersection). If one restricts to

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a certain class of graphs interesting connections between classes may arise. For example *dually chordal graphs* (graphs defined by a certain elimination scheme) are precisely clique graphs of chordal graphs, and *Helly chordal graphs* (clique-Helly chordal graphs) are precisely clique graphs of dually chordal graphs, cf. [5, 6]. Clique graph transformation has also been studied with respect to convergency to the one-vertex graph [2].

Bandelt and van de Vel drew attention to intersection graphs of maximal hypercubes with respect to *median graphs*. They noticed that the cube graph of an arbitrary median graph is *disk-Helly* [3], that is a graph in which disks enjoy the Helly property. Later Bandelt and Chepoi introduced *acyclic cubical complexes* in analogy with acyclic simplicial complexes [1]. They proved several characterizations of graphs of acyclic cubical complexes, analogies to characterizations of chordal graphs (note that the latter are precisely the graphs of acyclic simplicial complexes). In particular they showed that the cube graph of a graph of an acyclic cubical complex is dually chordal, as are the clique graphs of chordal graphs. They asked whether the converse is also true, that is whether every dually chordal graph is the cube graph of a graph of this class.

We will prove in Section 3 that the question of Bandelt and Chepoi has a positive answer. In the proof we use the concept of *expansion* which can be regarded as an elimination procedure (or more accurately its inverse). The expansion has first been used by Mulder in the characterization of median graphs [16], and Chepoi followed with a similar characterization of isometric subgraphs of hypercubes (*partial cubes* for short) [8]. Recently several classes of partial cubes have been introduced using the latter concept [7, 11]. The existence of an elimination procedure is an important feature of all the mentioned chordal-like classes, hence partial cubes are a natural environment to characterize these classes as cube graphs. Hence in Section 4 we introduce another class of partial cubes called *p-expansion graphs*, and we show that cube graphs of p-expansion graphs are precisely chordal graphs. In the proof we establish a connection with the tree for which a chordal graph is represented as the intersection graph of its subtrees. We continue with characterizations of Helly chordal graphs and *doubly chordal graphs* (graphs which are dually chordal and chordal) as cube graphs of two relevant classes of partial cubes. In the last section we prove that any graph is the cube graph of a bipartite graph, and briefly discuss some open problems.

2. Preliminaries

Throughout the paper we consider finite, simple, undirected graphs. A *neighbourhood* $N_G(x)$ is the set of neighbours of x in a graph G. A *closed neighbourhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. *Distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. Distance $d_G(U, V)$ between two subgraphs U, V of a graph is $\min\{d_G(u, v) : u \in U, v \in V\}$. A subgraph U of a graph G is *isometric* if $d_U(u, v) = d_G(u, v)$ for all $u, v \in U$. *Interval* $I_G(u, v)$ is the set of vertices on the shortest paths between u and v in G. A subgraph U of graph G is *convex* if $I_G(u, v) \subseteq U$ for all $u, v \in U$. (Indices in the above definitions are omitted when the graph is understood from the context.) Recall that a *hypercube* Q_k or a *k-cube* is the graph with vertex set $\{0, 1\}^k$

where two vertices are adjacent whenever they differ in exactly one position. In the k-cube the number k ($k \geq 0$) is called the *dimension* of a hypercube.

Partial cubes are isometric subgraphs of hypercubes. Important characterizations of partial cubes are due to Djoković [9] and Winkler [22]. We need the following sets in a graph G:

$$W_{ab} = \{x \in V(G) : d(x, a) < d(x, b)\}$$

$$F_{ab} = \{xy \in E(G) : x \in W_{ab}, y \in W_{ba}\}.$$

A bipartite graph G is a partial cube if and only if all sets W_{ab} are convex in G [9], and if and only if sets F_{ab} (for all $ab \in E(G)$) induce a partition of $E(G)$ [22]. This partition can be represented by the equivalence relation Θ as follows: edges ab, xy are in relation Θ if and only if $d(a, x) + d(b, y) \neq d(a, y) + d(b, x)$. Note that in a partial cube a path P is shortest if and only if no two edges on P are from the same Θ-class, cf. [12]. A graph G is *median* if for every triple of vertices $u, v, w \in V(G) : I(u, v) \cap I(u, w) \cap I(v, w)$ consists of precisely one vertex. It is well-known that median graphs are partial cubes [17] (see [13] for a recent survey on median graphs).

By $Q(G)$ we denote the *intersection graph of maximal hypercubes* of a graph G. That is, vertices of $Q(G)$ correspond to maximal (induced) hypercubes of G, and two vertices are adjacent if the corresponding hypercubes are nondisjoint. (In the case of partial cubes any hypercube is induced, thus for our purposes we may omit the condition about *induced* hypercubes. In a more general setting one might need to restrict to maximal *induced* hypercubes to avoid ambiguity). Given a vertex x of $Q(G)$ we shall denote by H_x the corresponding hypercube of G.

A *cubical complex* $\mathcal{K}$ is a set of (graphic) cubes closed for subcubes and nonempty intersections, cf. [1]. One may consider a complex $\mathcal{K}$ as a hypergraph with vertices representing 0-dimensional cubes, and edges representing cubes of $\mathcal{K}$ of larger dimensions. (For more on graphic cubes cf. [21], and for hypergraphs terminology cf. [4].) In the *graph of a cubical complex* $\mathcal{K}$ two vertices of $\mathcal{K}$ are adjacent whenever they constitute a 1-dimensional cube. A *cycle* of a complex $\mathcal{K}$ is $x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_1$ where x_i are distinct vertices and E_i distinct edges of $\mathcal{K}$, such that $x_i, x_{i+1} \in E_i$ for $i = 1, \ldots, k$ (mod k), and no member of $\mathcal{K}$ includes three distinct vertices of the cycle. Cubical complex $\mathcal{K}$ is *conformal* if any set of vertices, that are pairwisely in a common cube, is included in a member (that is in a cube) of $\mathcal{K}$. Complex $\mathcal{K}$ is *acyclic* if it is conformal and has no cycles. In this paper graphs of acyclic cubical complexes will be shortly called ACC graphs. Bandelt and Chepoi [1] observed that an ACC graph includes a *pendant hypercube* H, that is a hypercube for which there exists a hypercube R such that any intersection of H with other hypercubes is included in $H \cap R$ (note that H and R are necessarily maximal). They further showed that a graph is ACC if and only if it can be reduced to a hypercube by successive elimination of pendant hypercubes, where in each step all subcubes of a pendant hypercube that are not subcubes of other hypercubes are erased. Hence in each step of this elimination one gets an ACC graph with one less maximal hypercube. Furthermore ACC graphs are median, and a median graph is ACC if and only if it contains no convex bipartite wheels B_r for $r \geq 4$ (a bipartite wheel B_r is obtained from a cycle C_{2r} by adding a new vertex and edges between this vertex and every second vertex of the cycle).

Let G be a connected graph and G_1 its isometric subgraph. Then a *peripheral expansion* of G is a graph G' constructed as follows. Let G_1' be an isomorphic copy of G_1, and $G \cup G_1'$ a disjoint union of G and G_1'. For each vertex u of G_1 we denote the corresponding vertex in G_1' by u'. Now, G' is a graph obtained from $G \cup G_1'$ by adding edges between u and u' for all vertices $u \in V(G_1)$. We also say that we *expand* G_1 in G to obtain G'. This operation is a special case of expansion introduced by Chepoi [8], by which partial cubes are characterized as graphs that can be obtained by a sequence of expansions from K_1. Hence a graph obtained by a sequence of peripheral expansions from K_1 is a partial cube. Note that in each step of peripheral expansion one new Θ-class appears. Finally, median graphs are precisely graphs that can be obtained by a sequence of peripheral expansions from K_1 in which we expand a convex subgraph in each step [18].

3. Dually chordal graphs

We start with a new characterization of graphs of acyclic cubical complexes.

A subgraph H of a graph G forms a *cut* of G, if $G - H$ is a disconnected graph. Let $\mathcal{S}$ be a family of induced subgraphs (of a certain type) of a graph. A 2-*intersection* of $\mathcal{S}$ in G is a subgraph of G defined as the intersection of two distinct subgraphs from $\mathcal{S}$. In the following result we consider maximal 2-intersections of hypercubes (2-intersections of hypercubes that are not proper subgraphs of any 2-intersections of hypercubes). Note that if a 2-intersection is maximal then the corresponding two hypercubes are necessarily maximal hypercubes.

Proposition 1. *Let G be a graph of a cubical complex. Then the complex is acyclic if and only if every maximal* 2*-intersection of hypercubes forms a cut of G.*

Proof. Let G be a graph of an acyclic cubical complex. The proof is by induction on the number of vertices of G. If G is a k-cube then the claim of the theorem is trivially true. Otherwise G possesses a pendant cube H that is necessarily a maximal cube of G. By definition there exists another maximal cube R properly intersecting H such that any intersection of H with a distinct maximal hypercube of G is in $H \cap R$. Thus any maximal 2-intersection that involves H is equal to $H \cap R$ which obviously induces a cut. By removing all subcubes of H that are not in $H \cap R$ we obtain a smaller acyclic cubical complex in which maximal hypercubes correspond to the maximal hypercubes of G except for H. Moreover, there is an isomorphism between the smaller graph and the corresponding subgraph of G. By the induction hypothesis maximal 2-intersections form cuts in the smaller graph, hence they also form cuts in G.

For the converse let G be a graph of a cubical complex $\mathcal{K}$ such that any maximal 2-intersection of hypercubes forms a cut of G. Let H and R be any two maximal hypercubes such that $H \cap R$ is a maximal 2-intersection of hypercubes (if there are no such two hypercubes then G must be a hypercube, which is trivially an ACC graph). Thus $H \cap R$ forms a cut. Let $C_1, C_2, \ldots$ be connected components of $G - (H \cap R)$, and let C_1 be the component which includes some vertices of H. Let $\mathcal{K}'$ be a subcomplex of $\mathcal{K}$ consisting of hypercubes of $\mathcal{K}$ that have a nonempty intersection with C_1. Let $\mathcal{K}''$ be a complementary subcomplex of $\mathcal{K}'$ (that is, in $\mathcal{K}''$ there are maximal hypercubes of $\mathcal{K}$ that are not in $\mathcal{K}'$, along with all their subcubes). Obviously, maximal hypercubes of $\mathcal{K}$ are partitioned into those from $\mathcal{K}'$ and those from $\mathcal{K}''$, and so are the maximal 2-intersections (except $H \cap R$). We derive by induction on the number of maximal hypercubes that $\mathcal{K}'$ and $\mathcal{K}''$ are both acyclic. It is then easy to prove that $\mathcal{K}$ is also such. Indeed, note that the existence of a cycle in a cubical complex may be checked just within maximal hypercubes. Since any cycle in $\mathcal{K}$ should include vertices from both subcomplexes $\mathcal{K}'$ and $\mathcal{K}''$, it would have to include at least two vertices of H (or R), which would imply the existence of a cycle in $\mathcal{K}'$ (respectively $\mathcal{K}''$). Conformality of $\mathcal{K}$ follows from conformality of $\mathcal{K}'$ and $\mathcal{K}''$, thus $\mathcal{K}$ is acyclic. □

Using an analogous proof as above for simplicial complexes we derive the following result (note that any graph can be regarded as a graph of a simplicial complex).

Corollary 2. *A graph G is chordal if and only if any maximal* 2*-intersection of complete graphs forms a cut of G.*

Using the methods of the proof of Proposition 1 one can easily infer the following result which could be of some independent interest.

Corollary 3. *Let G be a graph of a cubical complex, and G not a hypercube. Then the complex is acyclic if and only if every maximal hypercube is either pendant or its vertices form a cut of G.*

In the proof of our main result we need a characterization of dually chordal graphs, due to Szwarcfiter and Bornstein [20]. A graph G is dually chordal if and only if G has a spanning tree T (called a *canonical tree*) such that for any two adjacent vertices x, y of G, vertices on the unique path in T between x and y form a complete subgraph in G. Alternatively, T is a canonical tree of G precisely when every closed neighbourhood of a vertex induces a subtree of T.

We will use the following easy lemma implicitly in the main theorem and in the sequel.

Lemma 4. *Let G be a partial cube, H a hypercube in G, and F_{ab} a Θ-class of G ($ab \in E(G)$). If H contains an edge of F_{ab} then every vertex of H is incident with an edge of $F_{ab} \cap H$.*

The following theorem answers the question from [1] showing that for every dually chordal graph G there is an ACC graph A such that $Q(A) = G$. In the proof an additional condition is used that connects some triangles of G with corresponding hypercubes of A. Note that if vertices u, v, w form a triangle in dually chordal graph G then in the canonical tree of G either one of the three vertices lies between the other two or there is another vertex z that lies between all the three vertices.

Theorem 5. *Let G be a dually chordal graph and let T be a canonical tree of G. Then there exists a graph A of an acyclic cubical complex with $Q(A) = G$ such that for any triangle of G with vertices $u, v, w \in V(G) : w \in I_T(u, v)$ if and only if $H_u \cap H_v \subset H_w$.*

Proof. The proof is by induction on the number of vertices of G. If $G \cong K_1$, then we set $A = K_1$, in which case the theorem is correct. Now, let G have more than one vertex, and let T be a canonical tree of G. Let x be a pendant vertex of T, adjacent to y. Note that $N_G[x] \subseteq N_G[y]$, because y is on the unique path in T from x to any vertex of T. By the induction hypothesis there is a graph A' of an acyclic cubical complex with $Q(A') = G - x$ such that the additional condition of the theorem holds for A' with respect to triangles of $G - x$ and tree $T - x$ which is clearly a canonical tree of $G - x$. The rest of the proof is the construction of a graph A from the graph A', such that $Q(A) = G$ and that the additional condition of the theorem holds for A with respect to triangles of G and canonical tree T.

Set $T' = N_G(x)$, and note that T' induces a subtree of $T - x$. Denote by $\mathcal{I}$ the family of maximal hypercubes of A' that correspond to vertices of T'. Let B be the subgraph of A' induced by the union of all hypercubes from $\mathcal{I}$. As every hypercube from $\mathcal{I}$ is nondisjoint with H_y, B is connected.

Claim. *Every hypercube in B is a subcube of a hypercube from $\mathcal{I}$.*

Proof of the Claim. Suppose there exists a hypercube H_z' in B that is not a subcube of any hypercube from $\mathcal{I}$. Let H_z be a maximal hypercube that includes H_z'. Clearly, H_z' intersects at least two hypercubes of $\mathcal{I}$. Let H_u be a hypercube of $\mathcal{I}$ such that $|H_u \cap H_z'|$ is maximum for all hypercubes of $\mathcal{I}$. Let F_{ab} be a Θ-class ($ab \in E(G)$) such that H_z' contains some edges of F_{ab}, while H_u does not contain any edge of F_{ab}. By Lemma 4 some of the vertices of H_u are incident with F_{ab} (because $H_u \cap H_z' \neq \emptyset$). Without loss of generality we may assume that $H_u \subseteq W_{ab}$. Then there exists a hypercube of $\mathcal{I}$ which contains some vertices of $H_z' \cap W_{ba}$, and as B is connected there is also such a hypercube which contains some edges of F_{ab}, let us denote it by H_v. By Lemma 4 we infer that H_v contains the corresponding edges of $F_{ab} \cap E(H_z')$, hence $H_u \cap H_v \cap H_z \neq \emptyset$. We also infer that none of $H_u \cap H_z'$, $H_v \cap H_z'$ is a subcube of the other (using also that $|H_u \cap H_z'|$ is maximum). Vertices u, v, z form a triangle in $G - x$, and either one of the three lies between the other two in $T - x$, or there exists a vertex w that lies on the shortest paths in $T - x$ between all pairs of u, v, z. In the first case, we derive by the additional condition of the theorem that only $z \in I_T(u, v)$ is possible (as only $H_u \cap H_v \subset H_z$ was not excluded), which is a contradiction with T' being connected. In the second case we derive by the additional condition of the theorem that $H_u \cap H_z \subset H_w$. As $|H_u \cap H_z'|$ is maximum for all hypercubes of $\mathcal{I}$ we derive that H_w is not in $\mathcal{I}$, which is again a contradiction with connectivity of T'. This concludes the proof of the claim. $\square$

The above claim in particular implies, that B is an induced subgraph of A' (as every edge induces a 1-cube which is in a hypercube of $\mathcal{I}$ by the claim). As B is an induced, connected subgraph, we can expand it from A' (in the definition of expansion we need B to be isometric in A' which will become clear later on; for the time being one can assume the natural extension of the definition of expansion to the induced, connected subgraphs). Thus let a graph A'' be obtained from a graph A' by the peripheral expansion in which B is expanded. By the claim above, no new maximal hypercubes arise in A''. Moreover, any hypercube H_u of A' either corresponds to H_u of A'' via isomorphism of A' to the subgraph of A'', or is the hypercube larger by one dimension than H_u in A'; in this sense we will speak of expanded and nonexpanded hypercubes in A''. Clearly, $Q(A'') = G - x$.

Now, let us prove that the additional condition of the theorem holds for A'' with respect to triangles of $G - x$ and the canonical tree $T - x$. Let $a, b, c \in V(G - x)$ form a triangle in G. First suppose that $a \in I_T(b, c)$. In A' we have $H_b \cap H_c \subset H_a$, and this is obvious also in A'', except in the case when both b and c are from T'. But then also $a \in T'$ so all three hypercubes are expanded and $H_b \cap H_c \subset H_a$ remains true in A''. For the converse, suppose that $H_b \cap H_c \subset H_a$ in A''. If they are all expanded hypercubes of A'' then we clearly have $H_b \cap H_c \subset H_a$ also in A' which in turn implies $a \in I_T(b, c)$. Suppose that two of them are expanded and one is not. If H_a were not expanded we would obviously have $H_b \cap H_c \subset H_a$ also in A' but this is a contradiction to T' being connected. Hence one of H_b or H_c was not expanded, thus again $H_b \cap H_c \subset H_a$ is true in A'. The remaining two cases are clear.

In particular, the additional condition of the theorem for A'' implies, that two hypercubes H_a, H_b form a maximal 2-intersection in A' if and only if the corresponding hypercubes form a maximal 2-intersection in A''. We will now prove that any maximal 2-intersection of hypercubes is a cut of A'' (that is, A'' is also an ACC graph).

Let H_a, H_b be hypercubes in A' that form a maximal 2-intersection. As A' is ACC they form a cut of A'. If both H_a, H_b are in $\mathcal{I}$, then it is clear that also in A'' expanded hypercubes form a cut. Now, assume that at least one of them (say H_a) is not from $\mathcal{I}$. Suppose that $H_a \cap H_b$ is not a cut in A''. Then there exists a path from $H_a \backslash H_b$ to $H_b \backslash H_a$ that must cross the expanded part of B. More precisely this path must cross the expanded part of $H_a \cup H_b$, otherwise $H_a \cap H_b$ would not be a cut in A'. Hence there exist two hypercubes H_u, H_v of $\mathcal{I}$ such that $H_u \cap H_v \cap H_a \cap H_b \neq \emptyset$ and $H_u \cap (H_a \backslash H_b) \neq \emptyset$, $H_v \cap (H_b \backslash H_a) \neq \emptyset$ (note that H_v can also be equal to H_b). But then $H_u \cap H_v \subset H_a$ (in A'), hence $a \in I_{T'}(u, v)$ by the additional condition of the theorem. Since H_a is not in $\mathcal{I}$, we derive that T' is not connected, a contradiction. Thus A'' is an ACC graph with $Q(A'') = G - x$, and the additional condition of the theorem holds for A''.

Now to the last step of the construction. In A'' denote by H'_y the expanded part of H_y, that is a subcube of H_y of which vertices appear in A'' but not yet in A'. Finally, in A'' expand H'_y, and call the resulting graph A. In A we obtain a new pendant cube which we call H_x. Note that H_x intersects precisely H'_y, and has the same dimension as H_y. Furthermore, for any hypercube H_u in A, the intersection $H_x \cap H_u$ is isomorphic to the intersection $H_y \cap H_u$ in A' (more precisely, $H_x \cap H_u$ is equal to $H'_y \cap H_u$). Hence in A the additional condition of the theorem for H_x in relation with other hypercubes follows from the relations of H_y with these hypercubes in A' (using that $u \in I_T(x, v)$ if and only if $u \in I_T(y, v)$ or $u = y$). Clearly, $Q(A) = G$, and A is an ACC graph because we just added a pendant hypercube to A''. The proof is complete. $\square$

4. Chordal graphs and their subclasses

In this section we introduce a class of so-called *p-expansion graphs*. Their definition is based on the (peripheral) expansion procedure in partial cubes which is a natural tool, similar to the simplicial vertex elimination scheme in chordal graphs.

1. K_1 is a p-expansion graph.
2. Every graph obtained from a p-expansion graph G by one of the following peripheral expansions is a p-expansion graph:

   a. expand an arbitrary vertex x of G;
   b. expand a union of maximal hypercubes $H_1, \ldots, H_k$ for which $\cap_1^k H_k \neq \emptyset$.

Let us first prove that p-expansion graphs are partial cubes. Obviously, K_1 is a partial cube, and the expansion in case 2.a is a peripheral expansion in which the isometric subgraph K_1 is expanded. We need to prove that the expanded subgraphs in case 2.b are isometric. We use induction on the number of expansion steps. Note that in a partial cube every path with no two edges from the same Θ-class is shortest [7, 12]. Let G be a partial cube, $H_1, \ldots, H_k$ maximal hypercubes in G with a nonempty common intersection, and $x, y \in \cup_{i=1}^k H_i$. If $x, y \in H_i$ for some i then obviously there is a path between vertices x, y in H_i with edges from different Θ-classes. Suppose $x \in H_i, y \in H_j, i \neq j$. Then let P_1 be a shortest path from x to $H_i \cap H_j$, and P_2 a shortest path from y to $H_i \cap H_j$. Moreover, as $H_i \cap H_j$ is a subcube of H_i (and of H_j), paths P_1 and P_2 contain no edges from Θ-classes of $H_i \cap H_j$. Now, by the induction hypothesis G is a partial cube, thus the common Θ-classes of H_i and H_j are precisely the Θ-classes of $H_i \cap H_j$ (to see this apply Lemma 4 and the transitivity of the relation Θ). Hence there is a shortest path from x to y in $H_i \cup H_j$, so $\cup_{i=1}^k H_i$ is the isometric subgraph of G. Therefore case 2.b is also a special case of peripheral expansion, hence p-expansion graphs are partial cubes. Moreover, as $\cap_{i=1}^k H_k \neq \emptyset$ and $\cup_{i=1}^k H_i$ is isometric we infer that no new maximal hypercubes appear in the expansion step 2.b (yet, a dimension of hypercubes $H_1, \ldots, H_k$ increases by one), and the intersection relations between them remain the same.

In the sequel we will use some well-known properties of chordal graphs (such as the simplicial vertex elimination scheme), cf. [10] or [14].

Proposition 6. *The cube graph of a p-expansion graph is chordal.*

Proof. Obviously $Q(K_1) = K_1$ which is chordal. We proceed by induction. By adding a pendant vertex in step 2.a we get a new maximal hypercube (of dimension 1) which intersects precisely a family of nondisjoint maximal hypercubes (possibly this family consists of only one maximal hypercube). That is, in $Q(G)$ we add a vertex and connect it to a complete subgraph. In other words, we add a simplicial vertex to a chordal graph, hence $Q(G)$ remains chordal. As observed above, in case 2.b neither the number of maximal hypercubes nor the intersection relations between them are changed. □

Conversely, we will prove a somewhat surprising result by connecting the p-expansion graph with a tree for which a chordal graph is represented as the intersection graph of some of its subtrees. Let $\mathcal{H}^*$ denote the collection of maximal hypercubes of G.

Theorem 7. *Let K be a chordal graph. Then there exists a p-expansion graph G such that $Q(G) = K$, and G has the following property. There exists a tree T in G, such that for any $H \in \mathcal{H}^*$, $H \cap T$ is a subtree of T, and for any $\mathcal{H} \subseteq \mathcal{H}^*$ we have: if $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$ then $\bigcap_{H \in \mathcal{H}} (H \cap T) \neq \emptyset$.*

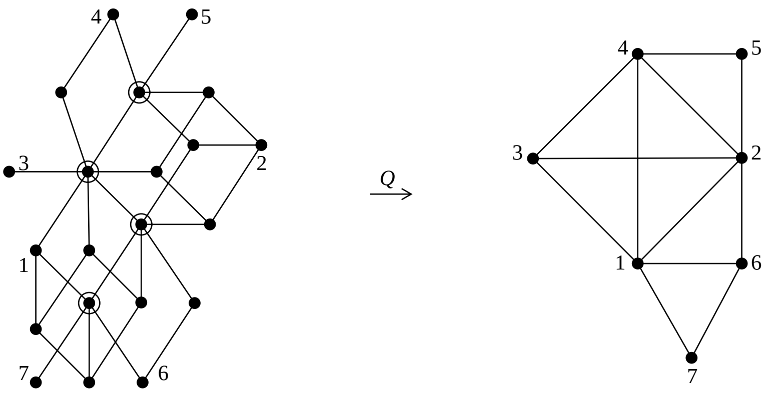

Fig. 1. p-expansion graph and its cube graph.

Therefore, T coincides with a tree for which K is the intersection graph of subtrees of a tree, and for each $H \in \mathcal{H}^*$, $H \cap T$ serves as the corresponding subtree.

Proof. The proof is by induction on the number of vertices of K. If K has only one vertex then the claim of the theorem is clear (note that then T should consist of this vertex). Otherwise let x be a simplicial vertex of K, that is $N(x) = \{x_1, \ldots, x_l\}$ induces a complete graph. Since $K - x$ is chordal, we may assume that there exists a p-expansion graph G' that has a tree T', and all assertions of the theorem hold for T', G' and $K - x$. Let $\mathcal{X} = \{H_{x_1}, \ldots, H_{x_l}\}$ be the family of hypercubes in G' that correspond to vertices of $N(x)$. Since vertices of $N(x)$ form a complete graph, we have $V = \cap_{i=1}^{l} H_{x_i} \neq \emptyset$ (this follows from the fact the maximal hypercubes in partial cubes enjoy the Helly property; alternatively one can deduce this through this proof for the case of p-expansion graphs by using induction). Hence, $V \cap T'$ is a nonempty subtree of T', and let $y \in V \cap T'$.

Suppose first that y does not belong to any hypercube from $\mathcal{H}^* \backslash \mathcal{X}$. Then add a pendant vertex z to be adjacent to y, and the construction is done. In this case we get a new maximal hypercube Q_1 with vertices y and z, $T = T'$, and $Q_1 \cap T$ is a subtree consisting of vertex y. Obviously, the last condition of the theorem remains correct in this case.

Now, suppose that y belongs to a maximal hypercube from $\mathcal{H}^* \backslash \mathcal{X}$. Then expand a subgraph $\cup_{i=1}^{l} H_{x_i}$. (Recall that neither the number of maximal hypercubes nor their intersection relations have changed by this expansion.) Let z be a neighbour of y obtained by this expansion. Set $T = T' \cup \{z\}$. Now, add a pendant vertex w to be adjacent to z, and call the obtained graph G. In G we get a new maximal hypercube Q_1 with vertices z and w, and $Q_1 \cap T$ is a subtree consisting of vertex z. Obviously $Q(G) = K$, and conditions of the theorem concerning tree T clearly remain true in G. $\square$

The example on Fig. 1 illustrates Theorem 7. On the right-hand side a chordal graph K is depicted with vertices labelled in the reversed order to the simplicial elimination scheme, that coincides with the order of adding maximal hypercubes to the corresponding p-expansion graph G. The latter is depicted on the left-hand side of the figure, where its maximal hypercubes are denoted by the same numbers as their corresponding vertices in K. Vertices of the corresponding tree T are circled (note that they induce a path P_4 in this example).

Helly chordal graphs and their subclass of doubly chordal graphs lie between chordal graphs and strongly chordal graphs, cf. [6]. Moscarini [15] considered them with respect to the algorithmic complexity of some well-known problems, and showed that they are NP-complete in the case of the Helly chordal graph, while efficient algorithms for doubly chordal graphs exist. These and related classes were studied extensively in [5].

Helly chordal graphs are chordal graphs in which cliques enjoy the Helly property (alternatively, they are chordal graphs in which disks enjoy the Helly property, since chordal graphs are dismantlable, cf. [2]). Bandelt and van de Vel observed that the cube graph of a median graph is Helly [3], so we infer

Corollary 8. *The cube graph of a median p-expansion graph is Helly chordal.*

Recall that doubly chordal graphs are precisely graphs that are both chordal and dually chordal [5, 15]. Thus

Corollary 9. *The cube graph of an ACC p-expansion graph is doubly chordal.*

In the following two theorems we prove converses of the two results above. The first theorem answers a special case of the problem raised by Bandelt and van de Vel, whether every disk-Helly graph is the cube graph of a median graph [3]. We show this in the case of chordal graphs.

In the proofs we will start with a p-expansion graph G, and prove that if $Q(G)$ is Helly (respectively dually chordal) then G must be median (respectively ACC). First a lemma.

Lemma 10. *Let G be a p-expansion graph constructed as in the proof of Theorem 7. Then, for any Θ-class F_{ab} with $k \geq 2$ maximal hypercubes $H_1, \ldots, H_k$ having edges in F_{ab}, there exist two maximal (disjoint) hypercubes $H' \subseteq W_{ab}$, $H'' \subseteq W_{ba}$, such that $H' \cap H_1 \cap \ldots \cap H_k \neq \emptyset$, $H'' \cap H_1 \cap \ldots \cap H_k \neq \emptyset$.*

Proof. Note that a new Θ-class appears by step 2.b precisely when we expand a union U of maximal hypercubes (having a common nonempty intersection), and add a pendant hypercube H' (isomorphic to Q_1) to the expanded part. Such an expansion is a result of the fact that there must exist a maximal hypercube H'' nondisjoint with all maximal hypercubes from U but disjoint with H'. Note that further expansions add edges in different Θ-classes, hence this does not change the fact that $H' \subseteq W_{ab}$, $H'' \subseteq W_{ba}$. The claim of the lemma follows. □

A *generalized sun* GS_n is the graph on $2n$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $x_1, \ldots, x_n$ form a complete subgraph, and vertices x_i, x_{i+1} are adjacent to y_i for $i = 1, \ldots, n \pmod n$, and edges $y_i y_{i+1}$ for $i = 1, \ldots, n \pmod n$ are optional (any subset (including empty) of these edges is allowed in GS_n). Vertices x_i are called *inner* and y_i *outer* vertices of a generalized sun. If outer vertices form a stable set then we will call this graph a *sun*, in symbol S_n. We say that a subgraph U is *dominated* in a graph G if there exists a vertex $x \in V(G)$ that is adjacent to all vertices of U. Recall that in median graphs convexity is equivalent to the weaker condition called 2-*convexity* [12]. An induced connected subgraph U is 2-*convex* in G if for any two vertices u, v of U such that $d_G(u, v) = 2$, every common neighbour of u and v belongs to U. A graph G satisfies the *quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with z a common neighbour of x and y, there exists a common neighbour v of x and y such that $d(u, v) = d(u, x) - 1$. It is well-known that median graphs satisfy the quadrangle property [13]. Also recall that a *clique* is a maximal complete subgraph of a graph.

Theorem 11. *Every Helly chordal graph is the cube graph of a median p-expansion graph.*

Proof. Let G be the p-expansion graph of which intersection graph $Q(G)$ is Helly chordal. We use the same construction as in the proof of Theorem 7. Recall (like in the proof of Lemma 10), that a new (nontrivial, i.e. with at least two hypercubes) Θ-class $F_e(e \in E(G))$ appears in G precisely when there exists a set of nondisjoint hypercubes $\mathcal{H} = \{H_1, \dots, H_k\}$ with a proper subset $\emptyset \neq \mathcal{H}' \subset \mathcal{H}$, such that H_i contains edges of F_e precisely when $H_i \in \mathcal{H}'$. Hence the new 1-cube is nondisjoint with hypercubes from $\mathcal{H}'$ and disjoint with hypercubes from $\mathcal{H} \backslash \mathcal{H}'$.

It is enough to prove that in every expansion step the subgraph U that we expand is 2-convex. Suppose that U is not 2-convex in some expansion step in which we build G (let the graph in this step be called G'). Then there exist two hypercubes $H_a, H_b \subseteq U$, and a hypercube $H_c \subsetneq U$ containing a 4-cycle C_c, such that $H_a \cap C_c = \{u, v\}$, $H_b \cap C_c = \{u, w\}$ and $H_a \cap H_b \cap C_c = \{u\}$. By the construction and Lemma 10 there exist three vertices in $Q(G)$: a neighbour ab of a and b (of which the corresponding hypercube appears in the last expansion step), a neighbour bc of b and c, and a neighbour ac of a and c such that a, b, c, ab, bc and ac form an induced GS_3 (some of ab, bc and ac could be adjacent, but a is not adjacent to bc, b is not adjacent to ac, and c is not adjacent to ab), and let us denote it by S. Note that S is an induced subgraph of $Q(G)$, even if this was not the last step by which we obtained G and $Q(G)$ (since any further steps just add simplicial vertices in $Q(G)$).

Since $Q(G)$ is a clique-Helly graph, there exists a vertex x that dominates S. In addition let x be chosen in such a way among vertices that dominate S that it appears in the largest number of cliques with the triple a, b, c. As x dominates a nonclique, its hypercube should have existed already in G', and as x is adjacent to ab, the corresponding hypercube H_x lies in U. Thus $H_x \not\supseteq C_c$, otherwise C_c would also lie in U. Since x is adjacent to all six vertices of S, H_x must possess edges of F_{uv} and F_{uw}. As $u \notin H_x$ (otherwise H_x would include C_c), there must be a Θ-class F_{uz} such that $F_{uz} = F_{vz} = F_{wz}$, and $H_x \subseteq W_{zu}$, while H_a, H_b, H_c have edges from F_{uz} (because they are nondisjoint with H_x). By Lemma 10 there is a hypercube H_y that is nondisjoint with H_a, H_b and H_c and $H_y \cap H_x = \emptyset$. In other words, there exists a clique in $Q(G')$ (and thus also in $Q(G)$) that includes vertices a, b, c and y such that y is not adjacent to x. Now, consider the following set of cliques: cliques with vertices a, b, ab, x; cliques with vertices b, c, bc, x; cliques with vertices a, c, ac, x; cliques with vertices a, b, c and y, and cliques with vertices a, b, c and x. All cliques are pairwisely nondisjoint as each of them possesses at least two vertices of $\{a, b, c\}$. As $Q(G)$ is Helly, their common intersection is nonempty, and let r be a vertex in the intersection.

Then r dominates S, and the number of cliques in which r lies with a, b, c is larger than the number of cliques in which x lies with a, b, c, a contradiction with the choice of x. Thus U must be 2-convex, and G is median. □

Lemma 12. *For every induced generalized sun S in a doubly chordal graph G there exists a vertex u in $G - S$ that dominates all inner vertices of S and three consecutive outer vertices of S.*

Proof. Let S be an induced generalized sun GS_n in a doubly chordal graph G. Observe that if there is an edge between any two outer vertices, this implies that we have an induced C_4 in G, thus G would not be chordal. Hence S is an induced sun S_n.

Let T be a canonical tree of G. Thus for any vertex $u \in V(G)$, $N_G[u] \cap V(T)$ induces a subtree of T. Consider $N_G[y_i] \cap V(T)$ where y_i are outer vertices of S for $i = 1, \ldots, n$. For each i, x_i and x_{i+1} (mod k) are in $N_G[y_i] \cap V(T)$, thus there is a path in T between x_i and x_{i+1}, which we call T_i and which is dominated by y_i. Hence the union $T_1 \cup T_2 \cdots \cup T_k$ induces a subtree of T such that vertices x_i are its leaves. As T is a canonical tree and x_i-s form a clique, all vertices of this subtree must be adjacent to all x_i-s. Hence all inner vertices x_i of S are dominated by all vertices of this subtree, and at least one vertex of this subtree is not from S. Now consider regions (faces) in the plane embedding of the subgraph induced by this subtree and by the cycle $x_1 \to y_1 \to x_2 \to y_2 \to \cdots \to x_k \to y_k \to x_1$. By the construction vertices on the perimeter of each region form a clique. Obviously, there is at least one vertex u from the subtree (not on the cycle) such that u is incident with three consecutive regions. □

Theorem 13. *Every doubly chordal graph is the cube graph of an ACC p-expansion graph.*

Proof. Let G be a p-expansion graph of which $Q(G)$ is dually chordal (and thus doubly chordal). We use the same construction of G as in the proof of Theorem 7. Since doubly chordal graphs are Helly [15], by Theorem 11, G is median. Thus by the result from [1] G is an ACC graph if it contains no convex bipartite wheels B_r for $r \geq 4$.

Suppose that G includes a convex subgraph U isomorphic to a bipartite wheel B_k for $k \geq 4$. Let $\mathcal{H} = \{H_{x_1}, H_{x_2}, \ldots, H_{x_k}\}$ be a set of maximal hypercubes such that $C_i \subseteq H_{x_i}$ where C_i are the 4-cycles of U. By Lemma 10 for each pair $H_{x_i}, H_{x_{i+1}}$ (for $i = 1, \ldots, k$ (mod k)) a hypercube H_{y_i} exists which is nondisjoint precisely with H_{x_i} and $H_{x_{i+1}}$ among hypercubes of $\mathcal{H}$. It is then clear that $x_1, \ldots, x_k, y_1 \ldots, y_k$ in $Q(G)$ form a subgraph S isomorphic to an induced generalized sun GS_k. By Lemma 12 there exists a vertex u in $Q(G) - S$ that dominates vertices $x_1, \ldots, x_k$ and three consecutive outer vertices (without loss of generality we can denote these three vertices by y_1, y_2, y_3). So in view of Lemma 10, H_u must include edges of all Θ-classes that appear in C_2 and C_3. Let F_{ab} and F_{bc} be the Θ-classes that appear in C_2, respectively C_3 (but not in both C_2 and C_3), where b is the central vertex of U. Since H_u is a hypercube, each of its vertices is incident with all Θ-classes that appear in H_u, so H_u must be disjoint with U (because U is convex). Hence there must be a Θ-class F_{bd} such that each vertex from C_2 and C_3 is incident with edges from F_{bd}, and H_u is in W_{db}.

Now, let b' be a vertex of H_u that is the closest to b of all the vertices of H_u. Let a' be its neighbour which is the closest to a among vertices of H_u, and similarly we set c' to be the closest to c in H_u. As H_u is a hypercube, a' and c' have two common neighbours, one of which is the vertex closest to b, that is b'. Let b'' be the other common neighbour of a' and c'. Clearly, $b'' \in W_{ab}$ and $b'' \in W_{cb}$, and $d(b'', a) = d(b'', c)$, hence we can apply the quadrangle property for vertices b'', a, c and b. It follows that a and c have a common neighbour distinct from b which contradicts the 2-convexity of U. Thus there are no convex bipartite wheels in G, so G is ACC. $\square$

5. Concluding remarks

1. It was mentioned in the Introduction that

Proposition 14. *Every graph is the cube graph of a bipartite graph.*

Proof. Let K be an arbitrary finite graph. We construct a bipartite graph G, such that $Q(G) = K$ as follows. To each vertex $x \in V(K)$ with $\deg(x) = k$, let H_x be the corresponding hypercube of dimension k, that is Q_k. Now, in each H_x choose a stable set of vertices $\{u_1, \ldots, u_k\}$ such that distances between them are even. This can be easily achieved, since in a hypercube precisely half of the vertices are from a stable bipartite set (hence the distances between them are even). Now, for each adjacent vertices a, b of K, identify two vertices $u_i \in H_a$ and $u_j \in H_b$ where the only thing for which we take care is that in every hypercube each vertex u_i is identified at most once. Because of the even distances between chosen vertices of each hypercube, G is bipartite. Note that any cycle that includes edges from different hypercubes (which correspond to vertices of K), has length at least 6. This means that no new hypercubes can arise in the process of identification of vertices and thus $Q(G) = K$. $\square$

2. The list of the known relations between graph classes:

Q(bipartite) = arbitrary
Q(p-expansion) = chordal
Q(median) $\subseteq$ Helly
Q(ACC) = dually chordal
Q(median p-expansion) = Helly chordal
Q(ACC p-expansion) = doubly chordal.

An open question (stated already in [3]): is Q(median) = Helly? To find other relations between certain bipartite classes and relevant nonbipartite classes via cube graph transformation seems to be an interesting challenge.

3. Results of this paper show connections between some classes of partial cubes and some classes of chordal graphs. Even a stronger analogy between the graphs of acyclic cubical complex and chordal graphs was shown in [1]. Namely, to a cubical complex $\mathcal{K}$ a simplicial complex $\mathcal{K}^{\Delta}$ was associated, the simplices of which are the nonempty subsets of cubes from $\mathcal{K}$. It is clear that the complex $\mathcal{K}$ is acyclic if and only if $\mathcal{K}^{\Delta}$ is acyclic. Following this approach it would be interesting to know whether there is such a connection

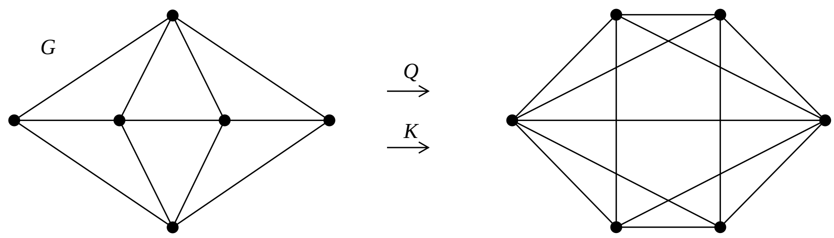

Fig. 2. Cube and clique graph of G coincide.

between p-expansion graphs and dually chordal graphs. Indeed, let $\mathcal{K}$ be a cubical complex of which the graph is a p-expansion graph. We associate it a hypergraph $\mathcal{K}^\nabla$ of which edges are the sets of maximal cubes of $\mathcal{K}$. It is clear that $\mathcal{K}^\nabla$ is a hypertree because it is Helly (hypercubes always enjoy the Helly property in partial cubes), and its line graph (in a hypergraph sense) which is a clique graph of a (2-section) graph of $\mathcal{K}^\nabla$, is chordal, cf. [4, 5]. In other words, the graph of $\mathcal{K}^\nabla$ is dually chordal. The natural question arises if the converse is also true. That is let $\mathcal{K}$ be a cubical complex. Is $\mathcal{K}^\nabla$ a hypertree only if the graph of $\mathcal{K}$ is p-expansion?

4. In the sense of hypergraphs, chordal graphs and dually chordal graphs arise as 2-section graphs of hypergraph classes which are each other's duals, notably acyclic hypergraphs and hypertrees. Furthermore, in a different sense acyclic cubical complexes are an analogous hypergraph class to acyclic simplicial complexes. Hence in this two-sided analogy a class of "cubical hypertrees" is missing. The question which hypergraph class this is may not necessarily have a unique and complete answer (since hypergraph duality between cubical complexes is not explicit as in the case of simplicial complexes). What is the role of p-expansion graphs and their corresponding hypergraph class in this context? The answer to the question in remark 3 above would already be a step forward in understanding these issues.

5. Let $L(G)$, $Q(G)$, $K(G)$ denote the line graph, respectively cube graph, respectively clique graph of a graph G. It is obvious that if G is a triangle-free graph then $L(G) = K(G)$; if G is C_4-free then $L(G) = Q(G)$; and if G is C_n-free (for $n \leq 4$) then $K(G) = Q(G)$. It seems to be interesting to characterize graphs that enjoy each one of these equalities. Observe that the above C_n-free ($n \leq 4$) condition is not necessary for $K(G) = Q(G)$. Consider the graph G obtained from $K_{2,4}$ in which we add three edges between vertices of degree 2 in such a way that they induce a path P_4, see Fig. 2. In both cases $K(G)$, $Q(G)$ is the graph on six vertices obtained from the complete graph by deletion of two nonincident edges. Two questions arise: are the above conditions on forbidden cycles enough in the cases $Q(G) = L(G)$ and $K(G) = L(G)$?

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